Diba and Grossman (1988) investigate whether rational bubbles exist in stock prices by developing a time series-based test focused on stationarity and cointegration properties of stock prices and dividends. Under rational expectations, stock prices should equal the discounted present value of expected future dividends, which implies that if dividends are nonstationary (e.g., contain a unit root), then stock prices should also be nonstationary but cointegrated with dividends—that is, their linear combination should be stationary. However, if a rational bubble is present—defined as an explosive, non-mean-reverting component in prices—this would break the cointegration between prices and dividends. To test this, they apply autocorrelation analysis and Dickey-Fuller unit root tests to stock price differences and the cointegrating relationship. Specifically, they follow Granger and Engle (1987) by estimating a cointegrating regression of stock prices on dividends and applying stationarity tests to the residuals of this regression. They further apply Bhargava tests and simulate artificial bubble series to confirm that their methodology can detect the presence of explosiveness when it exists. Similarly, Gallin (2005) examines whether U.S. house prices and rents share a stable long-run relationship by testing whether they are cointegrated—meaning that while both may follow unit roots individually, their linear combination (the rent-price ratio) should be stationary if housing markets are in equilibrium. Gallin employs a long-horizon regression framework—building on methods used by Campbell and Shiller (2001) in stock markets and Mark (1995) for exchange rates—using U.S. quarterly data from 1970:Q1 to 2003:Q4. He first applies Augmented Dickey-Fuller tests, finding that both prices and rents are nonstationary, but rejects the unit root in the rent-price ratio, concluding that prices and rents are cointegrated. In this case, the stationarity test is applied directly to the rent-price ratio, which effectively serves as the residual of a cointegrating relationship with an implicit (1, -1) cointegrating vector. To analyze dynamics of adjustment, Gallin regresses future three-year changes in rents and prices on the initial rent-price ratio and user cost of housing capital. Recognizing that standard long-horizon regressions may be biased when prices follow a unit root but not a strict random walk, Gallin constructs a bootstrap procedure that generates artificial data consistent with cointegration but under the null that rents alone correct. The results reject this null, indicating that both rents and prices adjust toward equilibrium, with prices adjusting more. Together, both papers demonstrate that cointegration testing is a powerful tool to detect the presence or absence of explosive components (bubbles) in time series: in Diba and Grossman, a break in cointegration signals bubbles, while in Gallin, sustained cointegration provides evidence against bubbles or persistent mispricing in housing.

Chan and Woo (2006) examine whether inflationary bubbles existed during four individual hyperinflation episodes—Germany, Hungary, Poland, and Russia—by applying cointegration-based bubble detection to each country's data separately. Following Engsted (1993, 2003) and based on the Cagan (1956) money demand model, they use two sequential cointegration tests: first, they regress real money balances (Mₜ - Pₜ) on expected inflation (ΔPₜ₊₁); second, they regress real money balances on money growth (ΔMₜ). According to theory, if no bubbles exist, these variables should be cointegrated (i.e., residuals stationary). In contrast, a bubble would break this cointegration in the second regression. The authors apply the Threshold Autoregressive (TAR) unit root test of Caner and Hansen (2001) to the residuals of these regressions to capture potential nonlinearities or regime shifts caused by periodically collapsing bubbles or monetary reforms—issues that standard linear cointegration tests (ADF, Phillips-Ouliaris Z, Gregory-Hansen) cannot handle. They also perform Monte Carlo simulations to confirm that TAR cointegration more reliably detects nonlinear bubbles. Their findings show stationary threshold cointegration in both regressions for all four countries, thus rejecting the presence of inflationary bubbles and validating the Cagan model. In sum, residual-based threshold cointegration was central to the methodology, with each country's time series tested individually for bubbles.

**In both cases, testing the stationarity of residuals (or their equivalent linear combinations) provides a direct and practical method to assess whether a time series exhibits explosive behavior or remains tied to its fundamental drivers.**

Phillips, Wu, and Yu (2011) develop an econometric methodology to detect and date explosive behavior in asset prices by introducing a recursive right-tailed unit root test based on forward-recursive Augmented Dickey-Fuller (ADF) regressions. The core idea is that under rational expectations, prices without bubbles should be cointegrated with dividends and non-explosive; if a bubble exists, prices will exhibit explosiveness (autoregressive root δ > 1), breaking cointegration with fundamentals. Unlike Diba and Grossman (1988), who used static cointegration tests, Phillips et al. apply recursive ADF regressions to log(Nasdaq real prices) and log(real dividends), calculating supremum ADF (SADF) statistics across forward-expanding subsamples, enabling date-stamping of bubble origination and collapse. Recognizing that traditional unit root and SADF tests lose power in the presence of multiple bubbles and collapsing-reinflating behavior (Evans, 1991), Phillips, Shi, and Yu (2015) extend this framework by proposing a Generalized sup ADF (GSADF) test that implements a double-recursive ADF procedure with flexible start and end points of regression windows. This enhances power to detect multiple distinct episodes of explosiveness. They further improve bubble date-stamping with a Backward sup ADF (BSADF) test, providing precise identification of bubble origination and termination dates. Validated through Monte Carlo simulations, this recursive unit root testing framework—augmented with GSADF and BSADF—offers a robust, flexible methodology for detecting and dating explosive dynamics and bubbles in financial time series.

Podhorsky (2024) studies whether the large price swings in Bitcoin represent true **bubbles**, or whether they can be explained by **fundamental value** based on **mining costs**. The author defines Bitcoin’s fundamental value as the **marginal cost of mining**, which depends on factors like mining difficulty, hardware efficiency, and electricity costs. To test if market prices reflect this fundamental value, Podhorsky first applies the **Engle-Granger cointegration method**: a regression of **Bitcoin prices on marginal mining costs** is estimated, and then the **residuals** are tested for **stationarity**. Finding that prices and mining costs are cointegrated suggests that Bitcoin prices are linked to fundamentals in the long run. To test for **short-run explosive behavior**, the author applies the **SADF test** (Phillips et al., 2011), first on the **raw Bitcoin price series**, which shows explosiveness, and then on the **residuals of the cointegration regression**. The key finding is that while raw prices appear explosive, the residuals do not — meaning the observed price booms are explained by shifts in fundamentals, not by speculative bubbles. This shows that combining **Engle-Granger cointegration** with **SADF testing** is an effective way to distinguish between **fundamentally-driven volatility** and actual **bubble behavior** in financial time series.

Horie and Yamamoto (2024) develop a methodology to detect and distinguish between **common** and **idiosyncratic bubbles** in large panel datasets. Instead of applying standard panel unit root tests to the raw data, they first decompose the panel using a **factor model**, where each time series is expressed as a combination of **common factors** (shared across all series) and **idiosyncratic components** (unique to each series). Their goal is to test whether bubbles are present in the common components, the idiosyncratic components, or both. To estimate the factor structure, they compare two methods: the traditional **PANIC approach** (Bai & Ng, 2004), which uses principal components from differenced data, and their preferred **Cross-Sectional (CS) method** (Yamamoto & Horie, 2022), which estimates the factor loadings from a "clean" pre-bubble period and then recursively reconstructs the common and idiosyncratic components over time. Once these components are extracted, they apply **SADF** and **BSADF tests** (Phillips et al., 2011; Phillips et al., 2015) to each component separately. The **SADF** test is used to detect the onset of explosive behavior by recursively applying right-tailed unit root tests, while the **BSADF** test helps accurately **date-stamp** the start and end points of any detected bubbles. Notably, they do not apply a **panel GSADF test** to the entire panel as done in some other studies (such as Potrykus, 2023); instead, they test each estimated component (common factors and idiosyncratic series) individually. This approach enables them to distinguish whether observed market-wide episodes (such as housing booms) are driven by national trends (common bubbles) or by localized factors (idiosyncratic bubbles).

Building on this recursive testing framework, Potrykus (2023) applies the **GSADF** and **BSADF** methodology of Phillips et al. (2015) to the **commodity markets**, extending its application from stock prices to a broad panel of commodity time series. Recognizing that most prior studies focused on **single commodities** or small groups, Potrykus systematically analyzes **35 commodities** divided into **three sectoral panels**: **energy** (coal, crude oil, natural gas), **metals** (10 industrial metals), and **agriculture & livestock** (22 commodities). Using **monthly data** from **1980–2021** (504 observations per series), the study first applies **GSADF and BSADF tests** to each **individual commodity**, detecting bubbles as periods where the test statistic exceeds Monte Carlo-derived critical values. To capture **cross-sectional dynamics** and **sectoral contagion**, Potrykus then implements **panel GSADF and panel BSADF tests** (Vasilopoulos et al., 2020), where the critical values are generated via **sieve bootstrap** to account for **error dependence across series**. This panel extension directly leverages the flexible **windowing of GSADF**, now applied not just to single time series but to **grouped commodity panels**, allowing the identification of both **individual** and **common sector-wide bubble episodes**. Results confirm that **energy and metals** sectors exhibit **more frequent and longer-lasting bubbles** than agriculture & livestock, and that **crisis events** (2007–2008 global financial crisis, 2010–2011 European debt crisis) triggered **synchronized bubble periods** across sectors. Thus, Potrykus demonstrates how the **GSADF/BSADF recursive testing framework**, extended to **well-defined commodity panels**, provides a powerful method for detecting and date-stamping **bubble dynamics** across **systems of related time series**.

Martínez-García and Grossman (2020) investigate whether **mildly explosive dynamics**—interpreted as housing bubbles—can be identified in international house prices using **recursive unit root testing techniques**. Their approach builds directly on the **sup ADF (SADF)** and **Generalized sup ADF (GSADF)** methodologies developed by **Phillips, Wu, and Yu (2011)** and **Phillips, Shi, and Yu (2015)**, which improve upon earlier static cointegration tests (such as Diba and Grossman, 1988) by enabling the **detection and date-stamping of multiple explosive episodes** in time series. The authors analyze a **balanced panel** of **23 countries**, using quarterly house price data from **1975Q1 to 2015Q4**. For the purpose of bubble detection, they do not apply a **panel GSADF test** across the panel; instead, they perform **country-by-country GSADF tests** on each individual house price series. The GSADF test involves **recursive right-tailed ADF regressions** conducted on moving windows of varying start and end points, which increases sensitivity to multiple bubble episodes. If the GSADF statistic exceeds simulated critical values, this indicates the presence of explosiveness. To precisely date the **origination and collapse** of bubbles, the authors further apply the **Backward sup ADF (BSADF)** test, following the procedure in **Phillips et al. (2015)**. This step produces a time series of **bubble periods** for each country. In summary, the paper adopts a **single-series recursive testing framework**, heavily influenced by the GSADF/BSADF literature, to detect and date explosive house price behavior across countries. While the study later uses **panel logit models** to analyze determinants of these bubbles, the core detection methodology is **univariate GSADF/BSADF applied separately to each country**, not a panel-based bubble test.